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Practice Midterm 1 October 13, 2008

1. (X, ρ) and (Y, σ) are metric spaces. Show that τ defined by

$$\tau\big((x, y), (x', y')\big) = \max\big(\rho(x, x'), \sigma(y, y')\big)$$

defines a metric on $X \times Y$.

Solution. Need to show non-degeneracy, symmetry, and triangle inequality.

(non-degeneracy)

Since ρ, σ are metrics on X, Y

$$\begin{aligned} \rho &\geq 0 \; \forall x, x' \in X, \text{ and } \sigma \geq 0 \; \forall y, y' \in Y \\ \implies \max(\rho, \sigma) &= \tau \geq 0. \\ \rho &= 0 \implies x = x' \\ \sigma &= 0 \implies y = y' \\ \implies \max(\rho, \sigma) &= \tau = 0 \implies x = x' \text{ and } y = y'. \end{aligned}$$

$\implies \tau$ non-degenerate.

(symmetry)

$$\begin{aligned} \tau\big((x, y), (x', y')\big) &= \max\big(\rho(x, x'), \sigma(y, y')\big) \\ &= \max\big(\rho(x', x), \sigma(y', y)\big) \text{ since } \rho, \sigma \text{ are metrics} \\ &= \tau\big((x', y'), (x, y)\big) \end{aligned}$$

$\implies \tau$ symmetric.

(triangle inequality)

Since ρ, σ are metrics

$$\begin{aligned} \rho(x_1, x_3) &\leq \rho(x_1, x_2) + \rho(x_2, x_3) \\ \sigma(y_1, y_3) &\leq \rho(y_1, y_2) + \rho(y_2, y_3). \end{aligned}$$

It follows that

$$\begin{aligned} \tau\big((x_1, y_1), (x_3, y_3)\big) &= \max\big(\rho(x_1, x_3), \sigma(y_1, y_3)\big) \\ &\leq \max\big(\rho(x_1, x_2) + \rho(x_2, x_3), \sigma(y_1, y_2) + \sigma(y_2, y_3)\big) \\ &\leq \max\big(\rho(x_1, x_2), \sigma(y_1, y_2)\big) + \max\big(\rho(x_2, x_3), \sigma(y_2, y_3)\big) \\ &= \tau\big((x_1, y_1), (x_2, y_2)\big) + \tau\big((x_2, y_2), (x_3, y_3)\big). \end{aligned}$$

$\implies \tau$ satisfies triangle inequality.

$\therefore \tau$ is a metric.

2. We define a map ϕ on sequences by $\{y_n\} = \phi\big(\{x_n\}\big)$ where

$$y_n = x_n x_{n+1}.$$

Show that ϕ is a continuous map from $l^2(\mathbb{R}, \mathbb{N})$ into $l^1(\mathbb{R}, \mathbb{N})$.

Solution. Let $x_n \to x \in l^2$, that is $||x_n - x||_2 < \delta$.

$$\begin{aligned}
||y_n - y||_1 &= \sum_{i=0}^{\infty} |y_i^{(n)} - y_i| \\
&= \sum_{i=0}^{\infty} x_i^{(n)} x_{i+1}^{(n)} - x_i x_{i+1}| \\
&= \sum_{i=0}^{\infty} x_i^{(n)} x_{i+1}^{(n)} - x_{i+1}^{(n)} x_i + x_{i+1}^{(n)} x_i + x_i x_{i+1}| \\
&= \sum_{i=0}^{\infty} |x_{i+1}^{(n)}(x_i^{(n)} - x_i) + x_i(x_{i+1}^{(n)} x_i)| \\
&\leq \sum_{i=0}^{\infty} |x_{i+1}^{(n)}(x_i^{(n)} - x_i)| + \sum_{i=0}^{\infty} |x_i(x_{i+1}^{(n)} - x_i)| \quad \text{Minkőwski's inequality} \\
&\leq ||x^{(n)}||_2 ||x^{(n)} - x||_2 + ||x||^2 ||x^{(n)} - x||_2 \quad \text{Hölder's inequality} \\
&< C\delta = \epsilon.
\end{aligned}$$

Therefore ϕ is continuous.

3. Show that, for all $\alpha > 0$, there is a constant C_α (independent of λ) such that

$$\int_0^{\infty} \frac{e^{-\lambda t}}{1+t} dt \leq \frac{C_\alpha}{\lambda^\alpha} \quad \forall \lambda > 0$$

Solution.

$$\begin{aligned}
\int_0^{\infty} \frac{e^{-\lambda t}}{1+t} dt &= \int_0^{\infty} \left| \frac{e^{-\lambda t}}{1+t} \right| dt \\
&= \left|\left| \frac{e^{-\lambda t}}{1+t} \right|\right|_1 \\
&\leq ||e^{-\lambda t}||_p ||1/(1+t)||_q \\
&= C \left(\int_0^{\infty} e^{-\lambda t p} dt \right)^{1/p} \\
&= C \left(-\frac{1}{\lambda p} e^{-\lambda p t} \Big|_0^{\infty} \right)^{1/p} = \frac{C}{\lambda^{1/p} p^{1/p}}.
\end{aligned}$$

Let $\alpha = 1/p \implies \int \leq \frac{C_\alpha}{\lambda^\alpha}$. This works for $0 < \alpha < 1$. For $\alpha > 1$. Make u-substitution

$u = \lambda t$.

$$\begin{aligned}\int_0^\infty \frac{e^{-\lambda t}}{1+t} dt &= \frac{1}{\lambda}\int_0^\infty \frac{e^{-u}}{1+u/\lambda} du \\ &= \int_0^\infty \frac{e^{-u}}{\lambda+u} du \\ &\leq C\left(\int_0^\infty \left(\frac{1}{\lambda+u}\right)^q du\right)^{1/q} \\ &= C\left(\frac{1}{p-1}\right)^{1/q} v^{\frac{1-p}{q}}\Big|_\infty^\lambda = C\left(\frac{1}{p-1}\right)^{1/q} \lambda^{\frac{1-p}{q}}.\end{aligned}$$

For $p = 1$, the integral does not converge. Let $\alpha = \frac{p-1}{q} \implies \int \leq \frac{C/\alpha}{\lambda^\alpha} = \frac{C_\alpha}{\lambda^\alpha}$.

4. Let $V = (\mathbb{R}^n, ||\cdot||_1)$ and $W = (\mathbb{R}^n, ||\cdot||_\infty)$. A is an $n \times n$ matrix. Find the induced norm for the linear map $\phi : V \to W$ given by $x \mapsto Ax$ (Note: V and W have different norms!)

   Solution.

$$||Ax||_\infty = \sup_i \left|\sum_{j=1}^n a_{ij}x_j\right| \leq \sup_i \sum_{j=1}^n = \sup_i ||a_{ij}||_\infty ||x_j||_1 = \sup_i \sup_j |a_{ij}|.$$

   To test, look at $n = 2$. Let $a_{k,l}$ be the max and choose e, basis vector that picks out the maximum value of each column.

$$||Ax||_\infty = \sup_j \left|\sum_{i=1}^n a_{ij}e\right|.$$

5. Show that there is a unique smooth solution to the *nonlinear* boundary value problem

$$u_x x = u^3, \qquad u(0) = u(1) = 0.$$

   Solution. Use energy norm

$$\left(\int_0^1 \frac{1}{2}u_x^2 dx\right)^{1/2}.$$

$$\begin{aligned}\int_0^1 uu_{xx}dx &= \int_0^1 u^4 dx \\ uu_x\Big|_0^1 - \int_0^1 u_x^2 dx &= \int_0^1 u^4 dx \\ -\int_0^1 u_x^2 dx &= \int_0^1 u^4 dx.\end{aligned}$$

But, both integrals are non-negative, so

$$\begin{aligned} \int_0^1 u_x^2 dx &= \int_0^1 u^4 dx = 0 \\ \implies ||u||_E &= 0. \end{aligned}$$

Assume v is another solution to the bvp. Then, $||u||_E + ||v||_E = 0$. This implies $||u - v||_E \leq 0$. But norm is non-negative, which implies $u = v$. So the solution is unique.